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1. REVIEWERS 2018
CHARACTERISTIC POINTS OF CONICS IN THE NET-LIKE METHOD OF CONSTRUCTION

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Abstract. The aim of this paper is to show how to complete the known net-like method for the case of a parabola or a hyperbola without using advanced methods of projective geometry. Only a construction of proportional segments is applied. Authors present a construction of the vertex of a parabola when its ideal point \( D^\infty \), a point \( B \), and a point \( A \) with the tangent \( t \) are given. In the case of a hyperbola defined by its vertices \( A \) and \( B \) and a point \( C \), the net-like method is completed by a construction of the hyperbola asymptotes. To understand the idea of this construction, a bit more complicated than the previous one, basic skills of elementary geometry, Pythagoras’ theorem and Thales’ theorem, are sufficient. In the case of a hyperbola defined by its asymptotes and a point, the presented construction of its vertices considering some parallelograms equal in area, follows from the well-known theorem about a line intersecting the hyperbola and its asymptotes.

Keywords: conics, parabola, hyperbola, ellipse, net-like methods, vertices of conics, asymptotes of a hyperbola, Pythagoras’ theorem, Thales’ theorem, proportional segments

1 Introduction
Former Descriptive Geometry programs included the basics of projective geometry. This was thus reflected in the classic textbooks for this subject (see [5], [6]). In those books, conic curves are defined and analyzed through projection transformations. Projective properties are used to formulate important theorems (Pascal’s and Brianchon’s) and to construct characteristic points.

Present course programs do not incorporate projective geometry. Therefore, we cannot consider conics as “products” of projection as E. Otto does in [5]. Nevertheless, the net-like method (see [3], p.142) resulting from this approach is presented to students as a way to construct points of an ellipse, parabola and hyperbola ([1], [3]). In order to achieve a satisfactory shape of these curves, it is of course better to know the characteristic points of these conics.

Diligent students using the CAD software are not always satisfied with the shape of the curve achieved by connecting through the “spline” command the consecutive points found by the net-like method. They accurately notice that perhaps the effect would have been better if characteristic points were among the constructed ones.

To use the CAD program to draw an ellipse, its vertices are necessary. This does not pose a problem, since affinity is part of the course, and these missing crucial points can be found by transforming the ellipse into a circle. It is impossible however, to do the same with a parabola or a hyperbola. When searching for the parabola/hyperbola vertices or asymptotes, students cannot transform them into a circle (as it once was standard – see [2] p. 120) because
central collineation is not part of the course program. There are also no practical exercises in applying the Pascal’s theorem, even if there are a few minutes during the lecture to mention it.

The aim of this article is to show how characteristic points may be constructed in the net-like method, relying only on the knowledge gained in high school (the ability to construct proportional segments).

2  The vertex of a parabola when its ideal point $D^\infty$, a point $A$ with the tangent $t$, and a point $B$ are given

In this case students know the net-like method of construction of points of the parabola in the form as presented in Figure 1. We will show that the construction of the parabola vertex can be based only on the construction of proportional segments, not new to students.

Label $d$ the line passing through $D^\infty$ and $B$, and $O$, the point of the intersection of $t$ and $d$ ; $|AO|= a$, $|BO|= b$. A point $P_i$ of the parabola is the point of intersection of two lines, $t_i$ and $d_i$. The line $d_i$ passes through $D^\infty$ and $X_i$, where $X_i$ lies on $t_i$ and is distant at $i \frac{a}{n}$ from $A$.

The line $t_i$ passes through $A$ and $Y_i$, where $Y_i$ lies on $d_i$ and is distant from $O$ at $i \frac{b}{n}$. By definition $x_i = i \frac{a}{n}$, $y_i = i \frac{b}{n}$, where $i$ is an integer. Therefore, for any “rational” point $P_i$ on the
parabola \( \frac{AX}{OY} = \frac{x}{y} = \frac{a}{b} \), there is no difficulty to generalize this result for any \( x_P \) and \( y_P \) defining a “real” point \( P \) on the parabola.

Hence, the construction of a missing point \( Q \) on the parabola consists in finding two segments \( AX_Q \) on \( t \) and \( OY_Q \) on \( d \) such that \( |AX_Q| : |OY_Q| = \frac{a}{b} \). The point \( Q \) is the point of intersection of \( t_Q \) and \( d_Q \), defined similarly as for a “rational” point \( P \). If one of them is given, the other can be found.

Using the Thales’ theorem that construction can be made as shown in Figure 2.

Figure 2 The construction of a missing point \( Q \) on the parabola

Figure 3 The construction of the point

Figure 4 The whole structure from figures 2 and 3
The vertex $V$ of the parabola is characterized by the fact that $d_V$ is the axis of symmetry of the parabola. Thus $d_V$ is the perpendicular bisector of the segment $AC$, where the point $C$ is symmetric on the parabola to $A$, lying on $t_C$ perpendicular to $d (AY_C \oplus d)$. Applying in Figure 3 the construction presented in Figure 2, one finds first $X_C$, next the line $d_C$ and the required point $C$.

The perpendicular bisector $d_V$ of $AC$ intersects $t$ at $X_V$, and $Y_V$ is found applying once more the construction from Figure 2. Figure 4 displays the whole construction.

**Remark**: the construction can be simplified. Once point $X_C$ is constructed, the line $d_V$ passes also through the midpoint $X_V$ of the segment $AX_C$.

### 3 The asymptotes of a hyperbola when its vertices $A$ and $B$, and a point $C$ are given.

Two perpendicular lines $x$ and $y$ intersecting at the given point $C$ are considered as number lines with zero points, $0_x$ and $0_y$ respectively, as it is shown in Figure 5. Let $X_i$ be the point on $x$ with the coordinate $x_i$, and $Y_i$ the point on $y$ with the coordinate $y_i$. Therefore, according to the net-like method, a "rational" point $P_i$ (lying on the hyperbola defined by $A$, $B$, and $C$) is determined as the intersection point of two lines, $a_i$ and $b_i$, where $a_i$ is passing through $A$ and $X_i$, and $b_i$ is passing through $B$ and $Y_i$, and $x_i = \frac{a_i}{n}$, $y_i = \frac{b_i}{n}$.

![Figure 5](image_url)

Figure 5 Two perpendicular lines $x$ and $y$ intersecting at the given point $C$ are considered as number lines with zero points, $0_x$ and $0_y$ respectively

Considering $n \to \infty$, one can describe a point $P$ on the hyperbola as common to two lines $a_P$ and $b_P$ such that $a_P$ is defined by $A$ and $X_P$ (with the coordinate $x_P$), $b_P$ is defined by $B$ and $Y_P$ on $y$ (with coordinate $y_P$), and $\frac{x_P}{y_P} = \frac{a}{b}$.

An asymptote $q$ of the hyperbola is passing through an ideal point $Q^\circ$. For $Q^\circ$ lying on the hyperbola, $a_Q \parallel b_Q$, with $\frac{x_Q}{y_Q} = \frac{a}{b}$ (Fig. 6).
Therefore, in order to determine the required asymptotes, one must find the coordinates $x_Q$ and $y_Q$ such that:

$$\frac{x_Q}{y_Q} = \frac{a}{b}; \tan \varphi = \frac{x_Q}{d} = \frac{a}{y_Q}, \text{ where } d = b+c. \tag{1}$$

From these relations we have:

$$x_Q^2 b = a^2 d. \tag{2}$$

After calculations we obtain a ratio:

$$\frac{x_Q}{a} = \frac{\sqrt{d}}{\sqrt{b}}. \tag{3}$$

The last proportion is not easy to construct directly. Because segments of the form $\sqrt{xy}$ can be constructed for any given segments $x$ and $y$ (see [4], p.18), we change the obtained equality into the following:

$$\frac{x_Q}{a} = \frac{\sqrt{ad}}{\sqrt{ab}}. \tag{4}$$

Now the construction of the required asymptote can be realized using the Thales’ theorem, as it is shown in Fig. 7.
The asymptote \( q \) (tangent to the hyperbola at the ideal point \( Q^\infty \)) is passing through \( Q^\infty \) and the midpoint \( O \) of the segment \( AB \). The other asymptote is symmetric to \( q \) with respect to the hyperbola axis (the line \( AB \)).

4 Vertices of a hyperbola when its asymptotes \( s \) and \( t \), and a point \( C \) are given

In the case of a hyperbola defined by its asymptotes and a point, in order to construct the missing points of the hyperbola one generally uses the method based on the following well known property (see for example [1], [3]):

I. *Segments of any line intersecting a hyperbola, included between the hyperbola and its asymptotes, are equal in length.*

This fact will be used to show another useful property of a hyperbola.

![Figure 7 The construction of the asymptote q by using the Thales' theorem](image)

![Figure 8 Illustration of: a) the property: two lines passing through a point on the parallelogram’s diagonal, parallel to the parallelogram’s sides, determine two parallelograms equal in area (crosshatched), b) the property II](image)

Notice that any parallelogram is divided by its diagonal onto two triangles equal in area. Thus, two lines passing through a point on the parallelogram’s diagonal, parallel to the
parallelogram’s sides, determine two parallelograms equal in area (crosshatched in Figure 8a). Consider now two points \( C \) and \( Q \) on a hyperbola, with its asymptotes \( s \) and \( t \) and lines passing through these points parallel to the asymptotes (see Figure 8b). As \( |C1| = |Q2| \) by the Property I, the triangles \( C13 \) and \( Q24 \) are congruent according to the criterion ASA (angle, side, angle). Consequently, the parallelograms \( O5C3 \) and \( O4Q6 \) are equal in area. Therefore, the following property is true as well.

II. Given a hyperbola with asymptotes intersecting at \( O \), parallelograms with sides parallel to the asymptotes, with one vertex at \( O \) and the other on the hyperbola, are equal in area.

Accordingly, consider now a hyperbola when its asymptotes \( s \) and \( t \) together with a point \( C \) are given. The parallelogram determined by \( C \) has the area equal to \( bcsina \). As a vertex \( A \) of a hyperbola defines a parallelogram with equal sides, therefore in order to construct it, one may find a segment \( a \) such that \( a^2sina = bcsina \), i.e. \( a^2 = bc \). The construction is shown in Figure 9.

Figure 9 Illustration of the construction of a segment \( a \)

Figure 10 The construction of points of a hyperbola defined by a point and asymptotes
The Property II allows us to determine a method (see Figure 10) of points of a hyperbola defined by a point and asymptotes, similar to that for equilateral hyperbolas (see [3], p.140).

References

PUNKTY CHARAKTERYSTYCZNE W SIATKOWYCH KONSTRUKCJACH UZUPEŁNIANIA PUNKTÓW STOŻKOWYCH

Celem tej pracy jest pokazanie jak uzupełnić metody siatkowej wyznaczania punktów hiperboli lub paraboli przez podanie konstrukcji punktów charakterystycznych tych krzywych, bez odwoływania się do zaawansowanych treści geometrii rzutowej. Autorki pokazują konstrukcję wierzchołka paraboli określonej przez dany kierunek $D$, punkt $C$, punkt $A$ ze styczną $t$. Wykorzystywana jest tylko konstrukcja odcinków proporcjonalnych. W przypadku hiperboli określonej przez dane wierzchołki $A$ i $B$ oraz punkt $C$ konstrukcja siatkowa jest uzupełniona o sposób wyznaczania asymptot tej hiperboli. Metoda jest nieco bardziej złożona niż w poprzednim przypadku, ale do jej zrozumienia także wystarcza znajomość geometrii elementarnej, twierdzeń Pitagorasa i Talesa. W przypadku hiperboli określonej przez dany jej punkt $C$ oraz asymptoty $s$ i $t$, podana konstrukcja jej wierzchołka, wykorzystująca tylko równość pól odpowiednich równoległoboków, opiera się na znanych twierdzeniach o odcinkach prostej przecinającej hiperbolę i jej asymptoty.